Naturalized Platonism
vs.
Platonized Naturalism*

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In this paper, we argue that our knowledge of abstract objects is consistent with naturalism. Naturalism is the realist ontology that recognizes only those objects required by the explanations of the natural sciences. But some abstract objects, such as mathematical objects and properties, are required for the proper philosophical account of scientific theories and scientific laws. This has led some naturalists to locate properties or sets (or both) in the causal order, and to suggest that philosophical claims about properties and sets are empirical, discovered *a posteriori*, and subject to

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revision. We call this view *Naturalized Platonism*, and in what follows, we contrast it with our own view, which we call *Platonized Naturalism*.[1]

Platonized Naturalism is the view that a more traditional kind of Platonism is consistent with naturalism. Traditional Platonism is the realist ontology that recognizes abstract objects, i.e., objects that are nonspatiotemporal and outside the causal order. The more traditional kind of Platonism that we defend, however, is distinguished by general comprehension principles that assert the existence of abstract objects. We shall argue that such comprehension principles are synthetic and are known *a priori*. Nevertheless, we claim they are consistent with naturalist standards of ontology, knowledge, and reference. Since we believe that Naturalized Platonism has gone wrong most clearly in the case of mathematics, we shall demonstrate our claims with respect to a comprehension principle that governs the domain in which mathematical objects, among other abstracta, will be located. This is the comprehension principle for abstract individuals, and in what follows, we show that our knowledge of mathematical truths is linked to our knowledge of this principle. Though we shall concentrate the argument of our paper on this particular principle, we believe that similar arguments apply to corresponding comprehension principles for properties, relations, and propositions.

I. Naturalized Platonism

Naturalized Platonism is an attempt to solve the problems inherent in traditional Platonism. One important problem concerns the very formulation of traditional Platonism. The problem is that traditional Platonists seem to rely on naive, often unstated, existence principles, such as that every predicate denotes a property (or picks out a class) or that a theoretical description of an abstract object is sufficient to identify it. But ever since Russell developed both his paradox of sets and his criticisms of Meinong, philosophers have recognized that such naive theories are often

[1]Different philosophers use the term 'naturalism' in different ways. See, for example, the anthology on naturalism by S. Wagner and R. Warner, eds., *Naturalism: A Critical Appraisal*, (Notre Dame, Ind: University of Notre Dame Press, 1993). D. Armstrong, in *Universals and Scientific Realism* (Cambridge: Cambridge University Press, 1978), reserves the term for the denial that there are any objects outside of spacetime and uses 'physicalism' to label the view that the natural world is physical. But we think that our definition of 'naturalism' is a serviceable one, in part because it contains a subtle ambiguity that reflects an ambiguity in the way in which the term is used in the literature. We shall say more about this ambiguity at the start of section VI.

fraught with contradictions and inconsistencies. And even if some formulation of Platonism proves to be clear and consistent, it would still face a second problem, namely, how we could ever know that the theory is true. Traditional Platonism seems to require that we have a mystical kind of cognitive access to entities outside the causal order by which we obtain knowledge of them. The logical positivists articulated this worry by arguing that our knowledge is either empirical or logical in nature and that in neither case could we have genuine, synthetic (ampliative) knowledge of nonspatiotemporal abstracta. For we can have empirical knowledge only of spatiotemporal objects, and logical knowledge is merely analytic. So, for the logical positivists, talk of abstract objects is just empty talk that arises from the mistake of reifying words into objects.

However, Quine suggested that some abstract objects (namely, sets and those mathematical entities thought to be reducible to sets) are on a par with the theoretical entities of natural science, for our best scientific theories quantify over both.[2] Quine formulated a limited and nontraditional kind of Platonism by proposing that set theory and logic are *continuous* with scientific theories, and that the theoretical framework as a whole is subject to empirical confirmation.[3] Because set theory and logic stand in the center of the theoretical web, they are isolated from immediate revision by their distance from empirical observations. Putnam modified Quine's view by arguing not simply that our best theories quantify over mathematical entities but that mathematics is *indispensable* to natural science (in the sense that there is no way to formulate such theories without quantifying over them).[4] Putnam also accepts properties on the grounds that they are needed in the proper formulation of natural laws.[5] Indeed, the appeal to properties also seems to provide a satisfying account of physical measurement, causal relations, biological functions, and inter-theoretic reduction.[6] On this conception, the acceptance of abstracta is constrained by principles of parsimony and reduction: (1) ontological commitment is to be kept to a minimum and governed by a small group of principles which are justified by the fact that they are essential to the workings of science, and (2) other purported abstract entities are to be reduced to sets (Quine) or sets and properties (Putnam).

[2] "On What There Is," reprinted in W. V. Quine, *From a Logical Point of View*, 2nd rev. ed. (Cambridge, MA: Harvard University Press, 1980), pp. 1-19.

[3] *Philosophy of Logic* (Englewood Cliffs, NJ: Prentice Hall, 1970).

[4] *Philosophy of Logic* (New York: Harper and Row, 1971); reprinted in H. Putnam, *Mathematics, Matter, and Method: Philosophical Papers I*, 2nd ed. (Cambridge: Cambridge University Press, 1979), pp. 323-357.

[5] "On Properties," reprinted in *Mathematics, Matter, and Method: Philosophical Papers I*, *op. cit.*, pp. 305-322.

[6] See C. Swoyer, "The Metaphysics of Measurement," in J. Forge, ed., *Measurement, Realism, and Objectivity* (Dordrecht: D. Reidel, 1987), for a description of some of the ways in which an appeal to properties clarifies our understanding of natural science.

Quine's formulation of a limited Platonism was seen by many as incomplete, however, for it did not provide an account of our access to abstract objects. How do we obtain *knowledge* of individual abstract objects? Gödel suggested that it was some perception-like intuition of those objects that guides our choice of axioms.[7] But Benacerraf pointed out that this is still not compatible with a naturalist theory of knowledge and reference.[8] Using the current causal theories of knowledge and reference as a guide, Benacerraf saw no natural way of linking our cognitive faculties with the objects known. And the problem persists even for the more recent externalist or reliabilist theories of knowledge, for how would one come to have reliable beliefs about nonspatiotemporal objects such as sets or properties? It is not clear how there could be reliable cognitive mechanisms for tracking and forming beliefs about such objects. Benacerraf also raised another question for Quine's limited Platonism, namely, how to arbitrate among equally acceptable reductions of other abstract objects to sets.[9] Benacerraf's principal example was the fact that the von Neumann ordinals and the Zermelo ordinals are just two (of infinitely many) equally viable ways of identifying the natural numbers with sets. There is no principled reason, therefore, to say that the numbers "really are" the von Neumann ordinals rather than the Zermelo ordinals, or vice versa.

Three trends have developed in response to the first of the Benacerraf problems we discussed: (1) Field[10] and Mundy[11] accept Benacerraf's problem as decisive and then challenge the idea that mathematics is indispensable to natural science. Mathematics may be useful, but only for representing features of the world that can be essentially characterized without an appeal to abstract individuals such as numbers or sets. Field,

[7] "What is Cantor's Continuum Problem," reprinted in P. Benacerraf and H. Putnam, eds., *Philosophy of Mathematics*, 2nd ed. (Cambridge: Cambridge University Press, 1983), pp. 470-485.

[8] "Mathematical Truth," *The Journal of Philosophy*, LXX/19 (November 1973): 661-679.

[9] "What Numbers Could Not Be," *Philosophical Review*, **74** (1965): 47-73.

[10] *Science Without Numbers* (Princeton: Princeton University Press, 1980).

[11] "Mathematical Physics and Elementary Logic," *Proceedings of the Philosophy of Science Association*, (1990): 289-301.

for example, abandons the attempt to naturalize platonism by abandoning platonism altogether. (2) Burgess simply rejects Benacerraf's problem as inapplicable to Quine's program, on the grounds that our beliefs about abstract objects are justified *as a whole* as part of our best scientific theories.[12] We simply don't need to justify our individual beliefs about particular abstract objects. (3) Armstrong[13] and Maddy[14] take Benacerraf's problem seriously and respond by more thoroughly naturalizing the entities in question. Armstrong locates properties within the causal order, and Maddy does the same for sets. The Benacerraf problem then simply dissolves, at least in the context of a naturalized theory of truth and reference such as that described by Field.[15]

Since our primary objective in this paper is to put forward our own positive view, we shall not rehearse in detail our reasons for not adopting one of these responses. However, it is important for us to sketch what we take to be their most serious *prima facie* problems, if only for the purpose of contrast with our own view. Of course, many of the points we raise in the remainder of this section have appeared in the literature. We begin with Field's view, even though he is not a Platonist, and so not a Naturalized Platonist. The main problems with his view are:

1. The complete dispensability of mathematics has not been established. It is doubtful whether the project can be carried out with respect to our most important physical theories, such as quantum mechanics and quantum field theory.

2. Even when mathematics is dispensable from actual science, it does not follow that it is dispensable from every scientific theory that might develop. We require an account of the language and subject matter of those portions of mathematics that might play a role in natural science, even if they don't currently play a role. And we even require an account of the dispensable portions of mathematics, if only to describe, as part of the very explanation of dispensability, the relation between the languages of natural science and mathematics.

3. Field uses the framework of second-order logic to show the dispensability of mathematics from classical physics. But he must reject the classical semantics of second-order quantifiers (as ranging over sets or properties). So his logic is developed using *modal* notions, and it is unclear whether those modal notions provide an adequate foundation for logic.

4. Field started his project by denying that numbers exist. But, what exactly is their status? Why is the language of number theory meaningful if its terms denote nothing at all? Field draws an analogy with fiction, claiming that '2+2=4' is true only in sense in which 'Holmes is a detective' is true.[16] But if numbers are useful fictions, then what is a fiction? No account is offered.

5. Recently, Field has suggested that numbers are abstract objects that happen not to exist. He accepts that they exist at other possible worlds. Field may have been led to this position for the following reasons. To explain the dispensability of mathematics, he attempts to establish its conservativeness, i.e., that there are no logical consequences of scientific theories involving mathematical claims that aren't already consequences of the nonmathematical portion of the theory. But recall that his notion of consequence is not the usual model-theoretic one, but rather modal. To figure out whether one claim follows from another, you have to consider a world in which the latter claim is true. So in order to talk about the consequences of scientific theories involving mathematical claims, one must consider worlds where the mathematical claims are true. In such worlds, the numbers exist. So Field is led to accept that numbers might have existed, but in fact don't. Yet if numbers don't in fact exist but might have, then what is the conception of contingently existing *abstract* objects that underlies this position? Why should abstract objects exist at some worlds and not at others?[17]

[12] "Epistemology and Nominalism," published in A. Irvine, ed., *Physicalism in Mathematics* (Dordrecht: Kluwer, 1990), pp. 1-16.

[13] *Universals and Scientific Realism*, *op. cit.*

[14] *Realism in Mathematics* (Oxford: Clarendon, 1990); and "The Roots of Contemporary Platonism," *Journal of Symbolic Logic*, **54**/4 (December 1989): 1121-1144.

[15] See H. Field, "Tarski's Theory of Truth," *The Journal of Philosophy*, LXIX (1972): 347-375. C. Swoyer and B. Mundy together constitute a variant of this Armstrong and Maddy camp—these philosophers deny the indispensability of sets but accept the indispensability of properties. They have a thoroughly naturalistic view of properties. See C. Swoyer, "The Metaphysics of Measurement," *op. cit.*, and B. Mundy, "The Metaphysics of Quantity," *Philosophical Studies*, **51** (January 1987): 29-54.

[16] *Realism, Mathematics, and Modality* (Oxford: Blackwell, 1989).

[17] This point is the subject of B. Hale and C. Wright, "Nominalism and the Contingency of Abstract Objects," *The Journal of Philosophy*, LXXXIX/3 (March 1992):

On the other hand, let us assume that Burgess is correct and that the Benacerraf problem has no force against Quine's limited kind of Platonism. Quine's view still faces certain other *prima facie* obstacles, however. The more serious ones are:

1. There is no account of mathematics that is not applied in scientific theories. Such mathematics certainly might be applied, and even if it is never applied, it is expressed in a meaningful language. How do we account for the meaningfulness of that language?

2. The mathematical portion of a scientific theory does *not* seem to receive confirmation from the empirical consequences derivable from the theory as a whole.[18] Sober points out that there is a core of mathematical principles common to all competing scientific hypotheses. Since this core group of mathematical principles are assumed in every competing theory, evidence for the theory as a whole confers no incremental confirmation on the purely mathematical portion.[19] Simply put, the evidence neither increases nor decreases the likelihood of those mathematical principles, since they are part of every competing hypothesis. This suggests that mathematics is not continuous with scientific theory.

3. If the overall scientific theory fails, scientists don't *revise* the mathematical portion but instead *switch* to a different mathematical theory. The revolutions in physics in the early part of this century were accompanied by appeal to previously unapplied mathematical theories of non-Euclidean geometries, not by revising Euclidean geometry. Even in those cases where the needs of physical theories spurred the development of new mathematics, those needs never altered the normal *a priori* procedures of mathematical justification by axiomatization, definition, and proof. This point also casts doubt on the continuity of mathematics with natural science.

4. The account of logic doesn't fit the facts. With the exception of quantum logic, no empirical evidence has ever been adduced in the course of arguing for alternative logics. The proliferation of alternative logics are not revisions of classical logic forced by empirical theory. Quantum logics stand alone, rather than as the first of a series of logics revised to suit the needs of physics.

5. The other problem posed by Benacerraf, concerning the arbitrariness of reductions, still remains. And even if other mathematical entities could be reduced to sets in a nonarbitrary way, it doesn't follow that they are just sets. Mathematicians who are not working on set theory do not take themselves to be studying sets. There is a strong intuition that every mathematical object is what it is and not some other (mathematical) thing.

Finally, we consider those philosophers who meet Benacerraf's challenge by more thoroughly naturalizing Platonic entities such as sets or properties (i.e., by locating them in the causal order). By accepting Quine's limited Platonism, Maddy inherits all of the problems just described (except for the first part of the last problem). But she and Armstrong face further difficulties as well:

1. For Maddy, there seems to be no way to assess the rationality of arguments for the highly theoretical axioms of ZF, such as the large cardinal axioms. This is the very part of the discipline that mathematicians find most interesting.

2. While Maddy solves the Benacerraf problem of arbitrary reductions by identifying numbers with structural properties of sets, the cost is that she denies the logical intuition (and common sense view of practicing mathematicians) that numbers are (individual) objects.

3. For Armstrong's sparse conception of properties and states of affairs, there is a problem of finding enough properties and states to account both for natural science and the mathematics it requires, without accepting uninstantiated properties.

111-135, and their followup article "A Reductio Ad Surdum? Field on the Contingency of Mathematical Objects," *Mind* **103**/410 (April 1994): 169-184. See also B. Linsky and E. Zalta, "In Defense of the Simplest Quantified Modal Logic," in J. Tomberlin, ed., *Philosophical Perspectives 8: Logic and Language* (Atascadero, CA: Ridgeview Press, 1994), pp. 431-458. In that paper, the present authors introduce contingently nonconcrete objects in order to give an "actualistic" interpretation of the simplest quantified modal logic (i.e., a logic that includes the Barcan formulas). But Field could not appeal to those objects to ground his conception, for numbers are necessarily, rather than contingently, nonconcrete.

[18] Indeed, even for the scientific portion of the theory, different pieces of evidence seem to bear on different parts of the theory. Confirmation doesn't seem to be holistic.

[19] "Mathematics and Indispensability," *The Philosophical Review*, **102** (1993): 35-57.

4. The combinatorial account of possibility Armstrong develops appeals to fictional entities (such as possible states of affairs), which don't seem to be part of the causal order.[20]

In the remainder of the paper, we develop an alternative that is free of these worries surrounding the various responses to Benacerraf's problem.

II. Platonized Naturalism

We motivate our view by reexamining the conception of abstract objects shared by both the traditional and naturalized Platonists. We believe that there are two mistakes in that conception: (i) the model of abstract objects as physical objects, and (ii) the piecemeal approach to theorizing about abstract objects. Once we are freed from these mistakes and get a proper conception and theory of abstract objects, answers to the apparent epistemological problems associated with Platonism quickly present themselves.

Most Platonists conceive of abstract objects on the model of physical objects. That is, they understand the objectivity and mind-independence of abstract objects by analogy with the following three features of physical objects: (1) Physical objects are subject to an appearance/reality distinction. This distinction can be unpacked in two ways: (a) the properties physical objects have can't be immediately inferred from the way they appear, nor can those properties be known in advance of empirical inquiry. Rather, they have to be discovered, and in the process of discovery we can be surprised by what we find. The fact that you think of a physical object as having certain features is no guarantee that it does. (b) There is more to a physical object than that presented to us by its appearances; for example, we assume that physical objects have "back sides". (2) Physical objects are sparse. You can assert that they exist only after you discover them. This means they have to be discovered in a piecemeal fashion, and this is sometimes guided by direct observation, sometimes guided by theoretical need. (3) Physical objects are complete. We simply assume that physical objects have all sorts of properties we may not know about (indeed, more properties than we could ever know about), and that they are determinate down to the last physical detail. So when we have a *bona fide* physical object x, then for every property F, either x has F or x has the negation of F. Features (1), (2), and (3) ground the objectivity and mind-independence of physical objects.

[20]See D. Armstrong, *A Combinatorial Theory of Possibility* (Cambridge: Cambridge University Press, 1989), especially pages 45-50.

We call those Platonists who conceive of and theorize about abstract objects on this model of physical objects *Piecemeal Platonists*. Historically, Piecemeal Platonism has been the dominant form of traditional Platonism, for traditional Platonists typically assume that their preferred abstract objects are "out there in a sparse way" waiting to be discovered and characterized by theories developed on a piecemeal basis. Naturalists are quite right to be suspicious of postulating causally disconnected abstract objects on a piecemeal basis, not simply because there is no explanation of how we can come to have reliable beliefs about them, but also because there seems to be no principled reason to accept some rather than others. If we are not differentially connected to abstract objects in some way, via some manifold analogous to spacetime, how could we come to have reliable beliefs about them, and how can we explain why certain particular abstracta exist while others don't? Though Quine offers a principled reason for accepting some abstract objects, he is a Piecemeal Platonist. He conceives of abstract objects on the model of physical objects, inheriting his conception from traditional Platonism. But we think that abstract objects are fundamentally different from physical objects, and that it is a mistake to conceive of them in this way. We see this model and the resulting piecemeal theories as the root of the apparent conflict between Platonism and naturalism. By rejecting this model, the essential compatibility of these two realist ontologies begins to emerge.

To explain the mind-independence and objectivity of causally inert abstract objects, one must assert topic-neutral comprehension principles that yield a *plenitude* of abstract objects. Comprehension principles are very general existence claims stating which conditions specify an object of a certain sort. Some of these principles are distinguished by the fact that they assert that there are as many abstract objects of a certain sort as there could possibly be (without logical inconsistency); i.e., some of these principles guarantee that the abstract objects in question constitute a plenum.[21] Any theory of abstract objects based on such compre-

[21]Some comprehension principles are unconditional; for example, a schema which requires, for every suitable condition, that there exists an abstract object that corresponds in some way to the condition. Others are conditional; for example, a modal conditional which asserts, for every suitable condition, that if it is possible that something satisfy the condition, then something exists that satisfies the condition.

hension principles constitutes a *Principled Platonism*. Some Principled Platonisms are built around comprehension principles for properties, relations and propositions.[22] However, in this paper, we appeal to the Principled Platonism formulated by one of the present authors, which in addition to comprehension principles for properties, relations, and propositions, and a comprehension principle for possibilia, includes a comprehension principle for abstract individuals.[23] The comprehension principle for abstract individuals will be the focus of our investigation, for it governs the domain in which mathematical objects will be located. We shall argue that a Principled Platonism and philosophy of mathematics based specifically on the comprehension principle for abstract individuals is consistent with naturalism. And, in our conclusion, we suggest that the argument extends to the other comprehension principles of this theory as well.[24]

Recently, other philosophers have suggested that a Platonism based on some sort of plenitude principles would account for the naturalist's epistemological concerns about mathematics. C. A. Anderson is a Platonist

[22]See, for example, N. Cocchiarella, "On the Logic of Nominalized Predicates and its Philosophical Interpretations," *Erkenntnis* **13** (1978): 339-369; T. Parsons, *Nonexistent Objects* (New Haven: Yale University Press, 1980); G. Bealer, *Quality and Concept* (Oxford: Oxford University Press, 1982); G. Chierchia and R. Turner, "Semantics and Property Theory," *Linguistics and Philosophy*, **11** (August 1985): 261-302; and C. Menzel, "A Complete, Type-Free 'Second Order' Logic and Its Philosophical Foundations," #CSLI-86-40 (Stanford: Center for the Study of Language and Information Press, 1986). Since these theories often place restrictions on the comprehension principles so as to avoid paradox, one might question whether they assert that there are as many universals as there could possibly be. However, within the framework of their respective theories, they yield as many universals as can be consistently added in a systematic way.

[23]See E. Zalta, *Intensional Logic and the Metaphysics of Intentionality* (Cambridge, MA: Bradford/MIT Press, 1988), and *Abstract Objects: An Introduction to Axiomatic Metaphysics* (Dordrecht: D. Reidel, 1983).

[24]The comprehension principles for properties, relations, and propositions formulated in Zalta (*ibid.*), like the ones mentioned in footnote 22, contain restrictions that prevent paradox (see footnote 33). However, in contrast to those other systems, the following principles are theorems of Zalta's system:

$$\Diamond\exists F\phi \rightarrow \exists F\Diamond\phi$$

$$\Diamond\exists F\, F{=}G \rightarrow \Box\exists F\, F{=}G$$

And in the type-theoretic formulation of the theory, it is axiomatic that if it is possible that a property exists, it does so necessarily (where, in this case, 'existence' is expressed by a predicate). These are all plenitude principles, for they ensure that there exist as many properties as there could be.

who informally suggests that every possible abstract object exists,[25] and M. Resnik's "postulational epistemology" seems to presuppose the idea that every possible pattern exists.[26] M. Balaguer argues that *if* a Platonist asserts that every possible abstract object exists, he or she can meet Benacerraf's epistemological challenge.[27] And in another context, namely, that of fiction, H. Deutsch has recently argued that the idea that there is a plenitude of fictional objects can reconcile platonism about fictional objects with the notion that authors create characters.[28] However, though these philosophers reject the sparseness of abstract objects, they still seem to conceive of abstract objects in terms of the other two elements of the model of physical objects. By constrast, we offer a different conception and a more detailed theory of abstract objects. As a result, we believe that we can develop a more general argument for the consistency of Platonism and naturalism.

In the remaining four sections of the paper, we present the three main components that distinguish our view: (1) a Platonism that is based on a comprehension principle for abstract individuals, (2) an analysis which locates mathematical objects in this ontology and a philosophy of mathematics based on this analysis, and (3) an argument that such a Principled Platonism is in fact consistent with naturalism. In section III and section IV, we present our version of Principled Platonism in enough detail to develop the analysis and philosophy of mathematics. Then, in section V, we address the epistemological underpinnings of this Principled Platonism and argue that it is consistent with naturalism. To anticipate briefly, the argument is that Principled Platonism is consistent with naturalism because such a Platonism is required to make sense of naturalistic theories; i.e., it is required for our very understanding of scientific theories. The comprehension principle and the logic in which it is framed are required for the proper analysis of natural language in general and mathematical language in particular. As such, they constitute the framework by which we make sense of any possible scientific theory. They therefore account for the way in which any possible scientific theory will be understood and

[25]"Logical Analysis and Natural Language: The Problem of Multiple Analyses," in P. Klein, ed., *Praktische Logik*, Göttingen: Vandenhoeck & Ruprecht, 1990, pp. 169-179.

[26]"A Naturalized Epistemology for a Platonist Mathematical Ontology," *Philosophica*, **43** (1989): 7-27.

[27]"A Platonist Epistemology," *Synthese*, **103**/3 (June 1995): 303-325.

[28]"The Creation Problem," *Topoi*, 10 (1991): 209-225.

thus for the very meaningfulness of any such theory. So we shall present a quasi-Kantian "transcendental" argument for the synthetic *a priori* character of the comprehension principle.[29] Taken together, the material in sections III though V constitutes Platonized Naturalism, which we put forward as an alternative to Naturalized Platonism. We then complete our argument in section VI by showing that Platonized Naturalism is indeed a kind of naturalism and is compatible with all the naturalist standards for ontology.

III. A PRINCIPLED PLATONISM

The theory of abstract objects to which we now turn does not appeal to any mathematical notions or axioms. However, both mathematical objects *and* mathematical theories will be identified as abstract individuals described by the theory.[30] The theory is expressed in terms of the predicate '$A!x$' (which asserts that *x is abstract*) and the primitive notion of *encoding*. Encoding is a mode of predication and, as such, contrasts with the traditional *exemplification* mode of predication. That is, in addition to the traditional reading of 'x is F' as x exemplifies F ('Fx'), we introduce the reading that x encodes F ('xF'). The three most important principles that govern the notion of encoding are:

1. For every condition on properties, there is an abstract individual that *encodes* exactly the properties satisfying the condition.
   $\exists x(A!x \,\&\, \forall F(xF \equiv \phi))$, where x is not free in ϕ

2. If x possibly encodes a property F, it does so necessarily.
   $\Diamond xF \rightarrow \Box xF$

[29] We say "quasi-Kantian" because, unlike Kant, we shall not ground the synthetic *a priori* character of the comprehension principles on facts about possible psychological states of consciousness and experience. See I. Kant, *Critique of Pure Reason*, Norman Kemp Smith, trans. (New York: St. Martins, 1965). Whereas Kant argued that the use of the categories of the understanding in judgements was a presupposition of any possible experience, we argue that the use of the abstract objects of our Principled Platonism is a presupposition of any possible science.

[30] A typed version of the theory asserts not only the existence of abstract individuals, but also the existence of higher-order abstract objects, such as abstract properties and relations (which are distinguished from ordinary properties and relations). See *Abstract Objects*, *op. cit*. Mathematical relations such as *successor* and *membership* can then be identified as abstract relations. But we shall not spend time at this point generalizing the theory to higher-order abstracta. So, in what follows, we use the term 'object' in the sense of 'individual'.

3. If x and y are abstract individuals, then they are identical iff they encode the same properties.
   $A!x \,\&\, A!y \rightarrow (x=y \equiv \forall F(xF \equiv yF))$

The first principle is the comprehension principle for abstract objects; the second principle says that what an abstract object (possibly) encodes is essentially encoded; the third principle is the identity principle for abstract objects. It is a simple consequence of the first and third principles that for every condition on properties, there is a unique abstract object that encodes just the properties satisfying the condition; there couldn't be two distinct abstract objects encoding exactly the properties satisfying a given condition if distinct abstract objects have to differ by at least one encoded property.

The comprehension principle asserts the existence of a wide variety of abstract objects, some of which are complete with respect to the properties they encode, while others are incomplete in this respect. For example, one instance of comprehension asserts there exists an abstract object that encodes just the properties Clinton exemplifies. This object is complete because Clinton either exemplifies F or exemplifies the negation of F, for every property F. Another instance of comprehension asserts that there is an abstract object that encodes just the two properties: *being blue* and *being round*. This object is incomplete because for every *other* property F, it encodes neither F nor the negation of F. But though abstract objects may be partial with respect to their encoded properties, they are all complete with respect to the properties they *exemplify*. In other words, the following principle of classical logic is preserved: for every object x and property F, either x exemplifies F or x exemplifies the negation of F. We can express this formally if we use λ-notation to define the negation of F ('$\bar{F}$') as follows:

$$\bar{F} \;=_{df}\; [\lambda y\, \neg Fy]$$

We may read the λ-predicate as: being an object y such that y fails to exemplify F. So we preserve the following formal principle of classical logic: $\forall F \forall x(Fx \vee \bar{F}x)$.[31]

The comprehension principle can be formulated without restrictions

[31] Note that encoding satisfies classical bivalence: $\forall F \forall x(xF \vee \neg xF)$. But the incompleteness of abstract objects is captured by the fact that the following is not in general true: $xF \vee x\bar{F}$.

because xF does not entail Fx.[32] It captures the idea that to describe any abstract object, one must specify a group of properties. But what is distinctive about abstract objects is that this is all one has to do to identify them completely. An abstract object encodes exactly the properties used to specify it. Another way of thinking about this is that the comprehension principle guarantees that no matter what properties one brings to mind to conceive of a thing, there is something that encodes just the properties involved in that conception. If properties are the possible ways of distinguishing among objects, and the comprehension principle asserts that there is an abstract object for every group of properties, then there are as many abstract objects as there could possibly be. So the comprehension principle asserts the existence of a plenitude of abstract objects.

Given the above, it should be clear that none of the elements of the model of physical objects apply to abstracta—abstract objects are simply different in kind from ordinary spatiotemporal objects. Ordinary spatiotemporal objects are not the kind of thing that could encode properties. Abstract objects are not the kind of thing that could be located in spacetime. We assert that abstract objects *necessarily fail to exemplify* certain ordinary properties. They necessarily fail to have a location in spacetime, they necessarily fail to have a shape, they necessarily fail to be material objects, they necessarily fail to be subject to the laws of generation and decay, etc. Consequently, by the classical laws of complex properties, abstract objects necessarily exemplify the negations of these properties. But notice that the properties abstract objects encode are more important than the properties they necessarily exemplify, since the former are the ones by which we individuate them. And it is important to mention that abstract objects may contingently exemplify certain relations to ordinary objects, such as being thought about by y, being studied by z, inspiring u to action, etc.

Our three principles of encoding are part of a larger system which includes complementary existence and identity principles for ordinary properties, relations, and propositions.[33] The framework as a whole has been

[32]So abstract objects may encode incompatible properties without contradiction, for the latter are defined as properties that couldn't be *exemplified* by the same objects. The following, for example, are jointly consistent: x encodes roundness (xR), x encodes squareness (xS), and necessarily everything that *exemplifies* being round fails to *exemplify* being square ($\Box\forall y(Ry \rightarrow \neg Sy)$). Thus, the notorious "round square" may simply be the abstract object that encodes just being round and being square.

[33]While the comprehension principle for abstract objects is not restricted, that for relations must be restricted to avoid paradox:

$\exists F^n \Box \forall x_1 \ldots \forall x_n (F^n x_1 \ldots x_n \equiv \phi)$, where ϕ has no encoding subformulas and no quantifiers binding relation variables

Identity conditions for relations can be defined in terms of the following identity conditions for properties: F and G are identical iff necessarily, all and only the objects that encode F encode G.

applied to the analysis of complex properties and propositions, possible worlds, and intentional entities such as fictions, among other things.[34] As another application of the theory, we now develop an analysis of mathematical theories and objects. Though we shall use formal number theory and set theory as typical examples of mathematical theories, our analysis covers informal mathematics as well. Informal mathematics, at the very least, provides stories about mathematical objects and such stories can be analyzed in the same terms we use to analyze mathematical theories.

We begin by extending the notion of an object encoding a property to that of an object encoding a proposition. We do this by treating propositions as 0-place properties. If we let 'p' range over propositions, then an object x may encode the proposition p in virtue of encoding the complex propositional property *being such that* p. We will symbolize such a propositional property as: $[\lambda y\, p]$.[35] These notions allow us to identify a mathematical theory T with that abstract object that encodes just the propositions asserted by T.

Next we define a technical notion of truth in a theory as follows: a proposition p is *true in* a theory T ('$T \models p$') iff T encodes the property *being such that* p. Formally:

$$T \models p \ =_{df} \ T[\lambda y\, p]$$

We use this definition to analyze the ordinary claim 'In theory T, a is F' as follows: the proposition that a exemplifies F is true in theory T, i.e., $T \models Fa$. In the present context, the symbol '$\models$' does *not* express model-theoretic consequence, but rather the family of notions such as true-at-a-world, factual-in-a-situation, and true-according-to-a-fiction, all of which have been analyzed elsewhere in terms of encoding.

[34]See E. Zalta, *Abstract Objects*, *op. cit.*, *Intensional Logic and the Metaphysics of Intentionality*, *op. cit.*, and "Twenty-Five Basic Theorems in Situation and World Theory," *Journal of Philosophical Logic*, **22** (1993): 385-428.

[35]The propositional property $[\lambda y\, p]$ is logically well-behaved despite the vacuously bound λ-variable y. It is constrained by the ordinary logic of complex predicates, which has the following consequence: x *exemplifies* $[\lambda y\, p]$ iff p, i.e., $[\lambda y\, p]x \equiv p$.

The final part of our analysis of mathematical theories is the Rule of Closure: mathematical theories are closed under the consequence relation. The rule is: if q is a consequence of $p_1, \ldots, p_n$, then if $p_1, \ldots, p_n$ are true in T, infer that q is true in T. Formally:[36]

$$\text{If } p_1, \ldots, p_n \vdash q, \text{ then from } T \models p_1, \ldots, T \models p_n, \text{ infer } T \models q$$

Notice that we may distinguish a theory from its axiomatization. The same theory can be axiomatized in different ways, for an axiomatization is just an initial group of propositions from which the theorems of the theory can be derived. But we are, in effect, identifying the theory with (the content of) its theorems, rather than with some particular axiomatization of it.

With this analysis of mathematical theories, we may now theoretically describe the mathematical objects of a given theory T in terms of the following analysis. Let κ be any constant or complex term (i.e., formed from primitive function symbols and constants) in the language of theory T. Then we say that the mathematical object κ of theory T ('κ_T') is that abstract object that encodes just the properties F such that, in theory T, κ_T exemplifies F.[37] We express this analysis formally as follows:

[36]Up to the statement of the present rule in terms of $\vdash$, we have reformulated the analysis in *Abstract Objects* (*op. cit.*, 147-153) in a way which more directly represents mathematical theories in terms of the primitive notion of encoding. But whereas the rule of closure in that work appealed to notion of necessary consequence, the present rule of closure replaces that with the notion of logical consequence, thereby correcting an error.

[37]We exempt the *natural* cardinals and *natural* sets from this analysis. These entities may be identified without an appeal to mathematical theories—their existence is a direct consequence of the comprehension principle. Let us say that properties F and G are *equinumerous* (with respect to the ordinary objects) just in case there is a relation R which is a one-to-one and onto function from the ordinary objects that exemplify F to the ordinary objects that exemplify G. Then, the comprehension principle asserts that for every property G, there is a unique abstract object x (call it *the number of Gs*) which encodes all and only the properties F equinumerous with G. A version of Hume's Principle, that the number of Fs is identical with the number of Gs iff there is a one-to-one correspondence between the (ordinary) Fs and the (ordinary) Gs, is now derivable from these definitions. A *natural cardinal* is therefore any object x such that for some G, x is the number of Gs. Frege's definition of *predecessor* and *natural number* can then be reconstructed, and given (a) the assumption that the formula defining *predecessor* denotes a relation and (b) the series of modal assumptions that 'there might have been n concrete objects' (where each assumption is expressed in terms of the numerical quantifiers), the five Peano axioms can be derived.

Similarly, a *natural set* is any object x such that for some G, x is the natural extension of G, where *the natural extension of* G is the unique abstract object y that encodes just the properties F materially equivalent to G (with respect to the ordinary objects). A consistent version of Frege's Basic Law V, that the extension of F is identical to the extension of G iff F and G are materially equivalent (with respect to the ordinary objects), is derivable from this definition. Moreover, if we say that an ordinary object z is a member of the natural set of Gs iff z exemplifies G, then the set-theoretic laws of extensionality, null set, pair set, and unions are derivable.

The technical details grounding these definitions and consequences may be found in E. Zalta, "The Theory of Fregean Logical Objects," unpublished manuscript. Copies may be obtained from the author by writing to the address CSLI/Cordura Hall, Stanford University, Stanford, CA 94305-4115 or extracted from cyberspace at the URL http://mally.stanford.edu/publications.html (under the heading Copyrighted Manuscripts).

$$\kappa_T = \imath y(A!y \;\&\; \forall F(yF \equiv \; T \models F\kappa_T))$$

For example, the number 1 of Peano Number Theory is the abstract object that encodes exactly those properties it exemplifies in Peano Number Theory. Similarly, the empty set $\emptyset$ of Zermelo-Fraenkel set theory is that abstract object that encodes just the properties it exemplifies in ZF. It is important to recognize that these are not *definitions* of the objects in question but rather theoretical descriptions. The descriptions are well-defined because we've established that for each condition on properties, there is a unique abstract object that encodes just the properties satisfying the condition. So the identity of the mathematical object in each case is ultimately secured by our ordinary mathematical judgements of the form: in theory T, a is F. We shall henceforth adopt the convention of translating the ordinary mathematical claim 'In theory T, a is F' into the language of our theory as '$T \models Fa_T$'.

It is an immediate consequence of this analysis that the claim that κ_T exemplifies F in T is equivalent to the claim that κ_T encodes F. Thus, the number 2 of Peano Number Theory (PNT) encodes the property of being prime iff 2_{PNT} exemplifies being prime in PNT:

$$2_{PNT}P \equiv PNT \models P2_{PNT}$$

Since encoding is a mode of predication, there is now a genuine sense in which ordinary sentences like '2 is prime' and '$\emptyset$ is an element of $\{\emptyset\}$' are true. Encoding predication is one reading of the copula, and provides the sense in which the ordinary sentence expresses a mathematical truth.[38]

[38]Compare Field (*Realism, Mathematics, and Modality*, *op. cit.*), who asserts that such sentences are false without offering a way of recapturing the intuition that they have a true reading. Field does draw an analogy between '2 is prime' and 'Holmes is

Given this reading, we can explain the *necessity* of ordinary mathematical statements by the fact that the encoding claims that provide the sense in which they are true are necessary. This is a consequence of the second principle of encoding.

We now have a conception of both abstract objects in general and mathematical objects in particular. This conception distinguishes the properties these objects encode from the properties they exemplify. On this conception, the properties attributed to a mathematical object in a theory are not the properties that it exemplifies *simpliciter*, for theories are frequently incomplete and inconsistent. On our view, mathematics is about the properties encoded by abstract mathematical objects. Mathematical objects certainly exemplify properties that are characteristic of their abstract nature, but the fact that they exemplify such properties is extra-mathematical. In the next section, we draw out consequences of this conception as we more fully describe the philosophy of mathematics which it serves to anchor.

IV. A Philosophy of Mathematics

By analyzing mathematical objects as *bona fide* abstract objects in a realist ontology, our Principled Platonism preserves the following traditional elements of Platonist philosophies of mathematics.[39] Mathematical objects are essentially different in kind from ordinary material objects. They are not spatiotemporal and therefore not subject to generation and decay. Mathematical truths are necessary, and moreover, mathematical objects necessarily exist.[40] Like all abstract objects, they couldn't possibly exemplify ordinary properties like having a shape, having a texture, being a building, etc. Indeed they necessarily exemplify the negations of these properties, and they contingently exemplify such properties as being denoted by a given symbol, or being thought about by mathematicians. Even though they encode only the properties attributed to them by their respective theories, they are nevertheless determinate objects, for they are complete with respect to the properties that they exemplify. Note that we preserve the common sense view that numbers are objects (i.e., individuals), and we preserve the logical intuition that the singular terms of mathematical theories denote abstract objects. Mathematical language is analyzed, from a logical point of view, in the simplest possible manner, and its semantics is therefore compositional. The truth conditions of mathematical sentences are stated in terms of the denotations of their terms and the way in which they are arranged.[41] This gives a sense in which the ontology is realist and in which its truths are objective.

Our view of theories and objects is very fine-grained. If the mathematical theories are different, the mathematical objects are different. Consider, for example, Euclidean, Riemannian, and Lobachevskian geometries. It is natural for a Platonist to think that different geometrical theories are about different objects. Moreover, Principled Platonism doesn't require us to reduce the various mathematical objects to the objects of some foundational theory. Each mathematical object is what it is and not some other thing. So Benacerraf's problem of explaining whether the Peano numbers "really are" the von Neumann ordinals or the Zermelo ordinals simply doesn't apply. Since Peano's theory of numbers is a different theory from the theory of Zermelo-Fraenkel sets, Peano numbers are not ZF sets of any kind.

It may be wondered whether our theory is too fine-grained, providing too many objects. The worry is that the number 1 of Peano's Number Theory seems to be the same object as the number 1 of real number theory, and that the emptyset of ZF is the same object as the emptyset of ZFC (i.e., ZF plus the Axiom of Choice). But we reply that these are not the same objects, and the reason they are not is that the number 1 of

a detective', and argues that both are acceptable only if prefixed by an 'In-the-story' operator. But he offers neither truth conditions for the story operator nor a reading of *unprefixed* sentences such as '2 is prime' on which they turn out true.

[39] For a summary of these traditional elements, see A. Irvine, "Introduction" to *Physicalism in Mathematics*, *op. cit.*, pp. xix–xx. Our theory also appears to preserve many of the aspects of historical Platonism. See J. Moravcsik, *Plato and Platonism* (Cambridge, MA: Blackwell, 1992).

[40] The necessary existence of abstract objects is a consequence of applying the Rule of Necessitation to the comprehension principle.

[41] For example, the sentence 'In Peano Number Theory, 3 is greater than 2' receives the analysis: $PNT \models 3 > 2$ (dropping the subscripts on '3' and '2'). The ordinary sentence '3 is greater than 2', inside the operator 'In Peano Number Theory', receives a traditional relational analysis of the form Rxy. Since theories are closed under logical consequence, it follows both that:

$$PNT \models [\lambda y\, y > 2]3$$

$$PNT \models [\lambda y\, 3 > y]2$$

So we may use the analysis of the previous section to identify the abstract objects denoted by 'Peano Number Theory', '2', and '3' in the compositional truth conditions for 'In Peano Number Theory, 3 is greater than 2'. This satisfies a desideratum Benacerraf described in 'Mathematical Truth,' *op. cit.*

Peano Number theory has (encodes) the property of being such that there is nothing between 1 and 2, whereas the number 1 of real number theory fails to have (encode) this property. So they are different. Similarly, $\emptyset_{ZF}$ and $\emptyset_{ZFC}$ are different because $\emptyset_{ZFC}$ has (encodes) the property of being such that every nonempty set of sets has a choice set, but $\emptyset_{ZF}$ lacks (fails to encode) this property. What this shows that that the objects are different because they are embedded in distinct theories.

We don't see mathematicians as searching for the unique true theory of sets. Consider a mathematician who at one time accepts the Continuum Hypothesis (CH), and then later rejects it, or consider two mathematicians who disagree about whether it is "true". We claim that they are thinking about different objects—they just don't realize it. Consider the analogous situation of a mathematician who at one time accepts the Axiom of Foundation and later "rejects" it, or the situation in which two mathematicians "disagree" about whether the Axiom of Foundation is "true". It seems clear in this latter case that the mathematicians are simply talking about different sets. The appearance of disagreement is explained by the common vocabulary. What each has in mind is perfectly real, but each party to the disagreement mistakes their limited portion of reality for the whole of reality. They can each be looking for the consequences of their own theories—there is an objective matter concerning whether a proposition is derivable from the axioms. But there is no single, hegemonic set-theory—there are many equally real universes of sets, and this applies not only to well-founded and non-wellfounded sets, but also to "CH" and "non-CH" sets. To suppose otherwise is to make the *mistake* of conceiving abstract objects on the model of physical objects—sets are not "out there in a sparse way" waiting to be discovered and described by one true theory. The real disagreements among set theorists concern the questions, which overall theory of sets is the most powerful and interesting, and which set theory is so powerful that we could have done without other mathematical theories?

Indeed, we extend our conclusion to claim that there is no single hegemonic membership relation. Our view is that not only should we not model abstract objects as physical objects, but we should not model abstract mathematical relations as ordinary relations. There is no appearance/reality distinction for mathematical relations; they are not complete or determinate. Rather, they are just the way we specify them to be—they are creatures of theory just as much as mathematical objects, and as

such, are indeterminate. So we treat mathematical relation R of theory T as that *abstract relation* that encodes just the properties of relations $\mathcal{F}$ that are attributed to R in T.[42] Thus, the membership relation of ZF can be theoretically described as follows:

$$\in_{ZF} \; = \; \imath R \forall \mathcal{F}(R\mathcal{F} \equiv ZF \models \mathcal{F}{\in_{ZF}})$$

If the theories are different, so are the abstract mathematical relations. To suppose otherwise is to make the mistake of modeling abstract mathematical relations as ordinary relations.[43]

Of course, by accepting an incorrect proof, a mathematician might erroneously judge that in theory T, x is F, for some T, x and F. The mathematical objects of a theory encode the properties that genuinely follow from that theory. It is possible to make a mistake about the properties that a mathematical object encodes by making a mistake about what properties follow from the theory. So we allow for error—a mistake about the objects of a theory is *not* a successful discovery of a truth about some different objects. Similarly, we allow for ignorance—mathematicians can form new judgements of the form 'In T, x is F' without thereby thinking of objects of a different theory. By allowing for error and ignorance, a version of the appearance/reality distinction presents itself in connection with our knowledge of mathematical objects. But this version of the distinction is rather different from the ones involved in the model of physical objects described earlier.

Nor is our theory a version of "if-thenism". If-thenism is the thesis that a mathematical claim like 2 is prime is really the claim that '2 is prime' is derivable from axioms of PNT, i.e., that apparent categorical assertions are really just certain logically true conditionals. But on our view, mathematical statements are categorical assertions. On our analysis, '2 is prime' is a simple categorical claim. Moreover, we distinguish the notion 'p is true in theory T' from the notion 'p is derivable from some axiomatization of T'. By collapsing the notions of truth in a theory and derivability in a theory, it would seem that if-thenists cannot maintain that a mathematical statement has a meaning of its own. For were it

[42] This requires the type-theoretic version of the comprehension principle for abstract objects mentioned in footnote 30. See *Abstract Objects*, *op. cit.*, Chapters V and VI.

[43] Note that, on our view, mathematical individuals are, in some sense, even more abstract than fictions. Whereas fictional individuals encode ordinary properties, mathematical individuals encode abstract properties.

to have one, someone could enquire about the truth of that statement independently of its derivability.

What is distinctive about mathematical objects is that they encode all and only their structural, mathematical properties. These are even more "essential" to them than properties which they necessarily exemplify, for their encoded properties are the ones we use to individuate them. This allows us to respond directly to Benacerraf's argument that numbers are not even objects, much less sets. Benacerraf says:[44]

> Therefore, numbers are not objects at all, because in giving the properties (that is, necessary and sufficient) of numbers you merely characterize an *abstract structure*—and the distinction lies in the fact that the 'elements' of the structure have no properties other than those relating them to other 'elements' of the same structure.

But on our analysis, the Peano numbers encode no properties that are superfluous to their role as numbers, yet they are objects nevertheless. Indeed, Maddy notes that Benacerraf's argument would not apply to objects whose numerical properties are the only ones they "have".[45] Our analysis captures the idea that the elements of a structure are *indeterminate* in some sense, defined solely by their structural relationships to other indeterminate elements of the structure. In effect, our abstract mathematical individuals objectify roles in structures. This, we think, satisfies the structuralist intuitions, without appealing to some undefined notions of structure, indeterminate element, or role, the existence conditions of which seem obscure.[46]

Nevertheless, we are able to take advantage of the structuralist account of the applicability of mathematics to the natural world. Physical science is successful in applying such nonspatiotemporal things to spatiotemporal objects because there are structural relationships between different

[44] "What Numbers Could Not Be," *op. cit.* (70).

[45] *Realism in Mathematics*, *op. cit.* (85).

[46] For various discussions of structuralism, see the previously cited works of P. Benacerraf and also the following: M. Resnik, "Mathematics as a Science of Patterns: Epistemology," *Nous* **16** (1982): 95-105, and "Mathematics as a Science of Patterns: Ontology and Reference," *Nous* **15** (1981): 529-50; S. Shapiro, "Structure and Ontology," *Philosophical Topics*, **17**/2 (1989): 145-72, and "Mathematics and Reality," *Philosophy of Science*, **50** (1983): 523-48; and M. Steiner, *Mathematical Knowledge* (Ithaca: Cornell University Press, 1975), and "Platonism and the Causal Theory of Knowledge," *The Journal of Philosophy*, LXX (1972): 57–66.

mathematical objects and features of the world; for example, those studied by measurement theory. Such relational notions as homomorphism relate objects in the world to abstract objects. It is part of the theory of abstract objects that ordinary objects and abstract objects can stand in (i.e., exemplify) relations to one another.

V. Principled Platonism, Reference, and Knowledge

Unlike ordinary objects, for which reference proceeds by some combination of causal processes, referential intentions and, perhaps, descriptive properties, reference to abstract objects is ultimately based on descriptions alone. Recall that it follows from the comprehension and identity principles for abstract objects that for every condition on properties there is a *unique* abstract object that encodes just the properties satisfying the condition. So the definite description $\imath x(A!x \ \&\ \forall F(xF \equiv \phi))$ is well-defined for each condition ϕ (with no free xs). There is, therefore, a straightforward account of reference to abstract objects. We analyze names of abstract objects in terms of these (rigid) descriptions.[47] Indeed, in section III, that is how we analyzed the names of mathematical objects.

This account avoids a problem facing those philosophers who accept a plenitude of abstract objects but who still conceive of them as complete objects that only exemplify their properties (on the model of physical objects). These philosophers can not account for reference to particular abstract objects (or account for our *de re* beliefs about them). The problem is that in the context of a plenitude of determinate abstract objects, (theoretical) definite descriptions are not well-defined. For example, 'the power set of omega' fails to denote because there are many different abstract objects that exemplify the property of being a power set of ω, each one different from the others with respect to one of the numerous other properties that fix its determinate nature (just consider all the various possible formulations of set theory and all the various objects in the possible models of a given set theory). By contrast, however, our conception

[47] We've indicated parenthetically that our descriptions of abstract objects are rigid designators. We introduce rigid definite descriptions into our formal language by semantic *fiat* (this follows *Abstract Objects*, *op. cit.*). The description $\imath x\phi$ is to be read as: the x which in fact is such that ϕ. This facilitates the technical development of the theory. Descriptions of a more conventional sort (i.e., non-rigid) can be added to the system, though all sorts of restrictions on substitution in modal contexts would be required.

introduces abstract objects that may be incomplete with respect to the properties that they encode, and this, together with the identity principle, ensures that incomplete descriptions will successfully refer.

Knowledge of particular abstract objects doesn't require any causal connection to them, but we know them on a one-to-one basis because *de re* knowledge of abstracta is by description. All one has to do to become so acquainted *de re* with an abstract object is to understand its descriptive, defining condition, for the properties that an abstract object encodes are precisely those expressed by their defining conditions.[48] So our cognitive faculty for acquiring knowledge of abstracta is simply the one we use to understand the comprehension principle. We therefore have an answer to Benacerraf's worry that no link between our cognitive faculties and abstract objects accounts for our knowledge of the latter. The comprehension and identity axioms of Principled Platonism are the link between our cognitive faculty of understanding and abstract objects.

The comprehension principle as a whole, we argue, is *synthetic* and known *a priori*. It is synthetic because it asserts the existence of objects encoding certain properties. It's not part of the meaning of 'abstract', 'encodes', and 'property' that for every condition on properties there is an abstract object that encodes just the properties satisfying the condition. So the principle isn't true in virtue of the very meanings of the words used to express it. Moreover, if it is known, it is known *a priori*. The reason is that it is not subject to confirmation or refutation on the basis of empirical evidence. This can be seen, at the very least, by inspection and analogy with the principles of logic—like such principles, no contingent facts bear on the truth of the comprehension principle. It is completely general, topic-neutral, and constitutes a simple extension of the *a priori* truths of logic.[49] Its *a priori* character can also be established, moreover, by the *a priori* character of its theorems.[50]

But the most important ingredient of Platonized Naturalism is the argument that Principled Platonism is consistent with naturalism. Indeed, from a naturalist perspective, how can there be synthetic *a priori* truths like the comprehension principle? The answer is that there can be such truths if they are required to make sense of naturalistic theories, that is, if they are required for our very understanding of those naturalistic theories. To establish that our comprehension principle is required in just this way, we offer a very general argument that begins with two premises. The first is that the logical framework required for the proper analysis of natural language and inference in general is the framework required for the proper analysis of scientific theories. In support of this premise, we simply note that scientific language is a special (if systematic) part of natural language, and that scientific inferences are analyzable in terms of logical consequence. Here we have in mind not simply the fact that the analysis of formal scientific theories will be based on the logical categories of *object* (*individual*), *property* (*quality*), and *exemplification* (*instantiation*), which play an essential role in the analysis of ordinary language, but also the fact that intensional language (such as talk about the future, talk about possibilities, subjunctive talk about what might have happened, the distinction between fact and fiction, etc.) shows up in scientific practice as a special case of the intensional language used in everyday discourse.[51] So the logical framework required for the analysis of natural language and inference as a whole is the framework that is required for the analysis of scientific theories and scientific reasoning. This first premise, then, simply expresses the idea that the laws of thought are, by their very nature, universally applicable, and that the analysis of scientific thought will be a special case of the analysis of thought in general. Now the second premise is simply that the comprehension principle and logic of encoding are an essential part of the logical framework required

[48]This depends, of course, on the fact that we can refer to the properties involved in the description. To explain such reference, we would rely on the theory of ordinary properties, the development of which would take us too far afield at this point. Briefly, however, the ordinary properties fall into two kinds, the ones denoted by primitive predicates and the ones denoted by complex predicates. With respect to the latter, reference to properties is by description, where the description is grounded by the comprehension principle for *properties*. With respect to the former, either reference to the property is by acquaintance (e.g., experienced, secondary qualities) or reference takes place in the context of a scientific theory (e.g., physical, primary qualities).

[49]This is similar to the turn-of-the-century logicians' view that the unrestricted comprehension principle for sets was known *a priori*. If one thinks that our (consistent) comprehension principle for abstracta just is logical rather than metaphysical, then our project can be seen as a kind of logicism.

[50]There is a wide range of *a priori* theorems that can be derived from the comprehension principle. See *Abstract Objects*, *op. cit.*, and "Twenty-Five Basic Theorems in Situation and World Theory," *op. cit.*

[51]In further support of this premise, we note that such fictions as frictionless planes, ideal gases, centers of gravity and other point-sized bits of matter, are used in scientific laws, and that the paradoxes of confirmation suggest that necessarily equivalent generalizations such as 'All ravens are black' and 'All non-black things are non-ravens' are distinct. Some charge that even the laws of science are useful fictions.

for the proper analysis of natural language and inference. In support of this premise, we claim that the comprehension principle and logic of encoding are the central components of an intensional logic that offers the best explanation of the logical form and consequences of such problematic constructions such as propositional attitude reports, modal contexts, discourse about fictions, puzzles about definite descriptions, and apparent failures of important logical principles.[52] We also point to our analysis in section III, which suggests that this logic also offers the best explanation of mathematical language and inference. From our two premises, we conclude that the comprehension principle and the logic of encoding are required for the proper analysis of scientific theories and inferences. As the final step to our argument, we claim that it follows from this conclusion that the comprehension principle and logic of encoding are required to make sense of any possible scientific theory, i.e., required for our very understanding of any such theory. So our Principled Platonism is consistent with naturalism because it is required by naturalism.

Note that the reason why Principled Platonism is consistent with naturalism is stronger than Quine's claim that his limited Platonism is part of the best scientific theory of the world. The claim that the comprehension principle is required for our understanding of any possible scientific theory is stronger than the claim that it is part of the best scientific theory. But we can accept Quine's talk about the best overall theory as long as we distinguish the notion of best overall *scientific* theory from the notion of best overall logical account of scientific theories. Arguments for the latter are different from the arguments for the former, for the latter is to be judged (i.e., justified or refuted) not by the empirical consequences derivable from one specific theory, but rather by the way the framework offers a uniform understanding of the variety of different scientific theories (among other things). Indeed, on the basis of such arguments, we can reasonably claim to know (i.e., to be justified in believing) the comprehension principle. For if on the basis of rational argument we conclude that our particular comprehension principle is part of the best logic and offers the most uniform understanding of scientific theories, then we are justified in believing it. We emphasize that we may be wrong about which framework offers the most uniform understanding of thought and language, in general, and of mathematics, in particular. Some other Principled Platonism may have the best analysis of logical form (and if so, the ideas in the previous paragraph would apply to its comprehension principle rather than ours). Though we claim that our comprehension principle is *a priori*, there is room for rational debate about its status as part of the best overall framework. We can be fallibilists about the *a priori*.

[52]See *Intensional Logic and the Metaphysics of Intentionality*, *op. cit.*

So we can claim to know the comprehension principle if we can rationally conclude that it is part of the best analysis and offers the most uniform understanding of scientific theories. To this end, we note that our analysis of mathematics covers the whole range of (possible) mathematical theories. Each consistent such theory could have been used in some possible world in one of the natural sciences there. Every consistent mathematical theory describes an abstract mathematical realm that, however bizarre or convoluted, might be needed to characterize some portion of the physical reality of some metaphysically possible world. This is why it is important to have an analysis of currently unapplied or dispensable mathematical theories. By offering a correct representation of any such mathematical theory, our logic and ontology makes sense of any possible science in which that theory is employed. Indeed, the full comprehension principle must be accepted for this project. Only a maximally broad comprehension principle is sufficient to the task of providing an analysis of every possible mathematical theory employed in a possible science (anyone claiming that a weaker theory could do this job has to explain why their theory, which would have to be more complex than our single principle, has not made some arbitrary choices about which mathematics might prove useful). So some such maximal comprehension principle is required for the analysis of mathematics, and given both that our specific principle suffices for this task and that there is currently an absence of alternatives that are sufficient to the task, we believe ours to offer the best explanation.

We conclude this section with a brief discussion of mathematical knowledge. The truths of mathematics, on our view, inherit the synthetic *a priori* character of the comprehension principle. The basic truths of mathematics are not such unadorned sentences as '2 is prime' and '$\emptyset \in \{\emptyset\}$', but rather have the form:[53]

In Peano Number Theory, 2 is prime.

[53]There are, however, some unadorned claims about the properties that the *natural* cardinals and the *natural* sets encode. These are independent of any mathematical theory and are derivable from our comprehension principle. See footnote 37 for a brief sketch.

In Zermelo-Fraenkel Set Theory, $\emptyset \in \{\emptyset\}$.

These are analyzed as encoding claims (dropping subscripts on the terms):

$PNT \models P2$
$ZF \models \emptyset \in \{\emptyset\}$

On our analysis, the truth of these encoding claims ultimately derives from the comprehension principle. So they are synthetic *a priori* and no further justification is needed. Note that these claims also have the appearance of analytic truths since their derivation appeals to the analyses of the respective theories. Truths of this kind will be discovered either by discovering whether the proposition in question is derivable in the theory or by identifying a new theory. It is a general feature of our metaphysical foundations that it represents meanings as objects. So the analysis of meanings consists of discovering synthetic truths about objects.[54] We may therefore return to the original notion of analysis on which it is objects that are analyzed, rather than sentences and/or predicates.

VI. Platonized Naturalism and Naturalist Standards of Ontology

We have now assembled the essential components of Platonized Naturalism: a Principled Platonism, an analysis and philosophy of mathematics, and an argument that Principled Platonism is consistent with naturalism. Platonized Naturalism is indeed a kind of naturalism, at least in the following sense. We defined naturalism at the outset, somewhat ambiguously, as the realist ontology that recognizes only those objects required by the explanations of the natural sciences. Platonized Naturalism satisfies this definition (in two senses) because it only recognizes the objects falling under the quantifiers of scientific theories *and* the objects required for a proper philosophical account of those theories. Unlike Naturalized Platonism, Platonized Naturalism postulates no new and novel occupants of spacetime, such as naturalized sets or immanent universals, or novel cognitive mechanisms for tracking such objects, such as Maddy's "set-detectors".[55] It postulates nothing in spacetime beyond the scope of natural science. Indeed, Platonized Naturalism leaves all our intuitions about the natural world intact. The objects of the natural world are still mind-independent, objective, and sparse, and the truths about them are discovered *a posteriori*.

[54]For a similar view about how analytic truths can also be viewed as synthetic, see C. A. Anderson, "Logical Analysis and Natural Language," *op. cit.* J. J. Katz, in *Language and Other Abstract Objects* (Totowa, NJ: Rowman and Littlefield, 1981), has also developed a theory that treats meanings as abstract objects.

[55]P. Maddy, *Realism in Mathematics*, *op. cit.* (65).

Moreover, Platonized Naturalism postulates nothing outside spacetime that could be subject to enquiry by one of the natural sciences. This is because it doesn't conceive of abstract objects on the model of physical objects. Abstracts objects are not "out there in a sparse way" waiting to be discovered on a piecemeal basis. The mind-independence and objectivity of abstract objects is explained by the objective truth of the comprehension principle. This is just a consequence of straightforward realism about the principle. Indeed, we now argue that in asserting this principle, we have satisfied the very standards of simplicity and parsimony that led naturalists to reject abstract objects in the first place.

One reason that Platonized Naturalism is simple is that a single, formally precise principle asserts the existence of all the abstract objects there could possibly be. The comprehension and identity principles present us with an ordered realm of objects that looks more like a formal garden than a jungle. A second reason that Platonized Naturalism is simple is that it postulates abstract objects in a nonarbitrary, nonpiecemeal way. One of the standards of natural science is to avoid introducing new kinds of theoretical entities in an arbitrary way, without any characteristic body of evidence for them. A principle asserting the existence of a plenitude of abstracta meets this standard, for a plenitude is not arbitrary. There is no need to explain why some abstract objects exist while others don't. Naturalists have rejected traditional Platonism, in part, because it is arbitrary in this respect—there seems to be no way to settle ontological disputes among traditional Platonists. It is also a naturalist standard that for each object discovered piecemeal in the causal order, you must give an account of our knowledge of its particular existence. A Platonized Naturalist, however, operates in the spirit of this standard by pointing out that for objects outside the causal order, there is no way to discover or explain our connection to them on such a piecemeal basis. There is no good reason to suppose that abstract objects are sparse.

Indeed, these ideas suggest that Platonized Naturalism is ontologically parsimonious. To satisfy the constraints of ontological parsimony, one should add as few objects as possible in a non-arbitrary way. But with abstract objects, the only way to add as few objects as possible in a non-arbitrary way is to add them all! The traditional justification for citing Ockham's Razor is that by keeping the number of kinds of theoretical entities to a minimum, theories are kept simple. Platonized Naturalism acknowledges that a maximal ontology of abstracta is the simplest because a plenum is not an arbitrary selection from some larger class.

Our comprehension principle is consistent, so it is contradiction and paradox free.[56] But some Naturalized Platonists have objected to comprehension principles for abstract objects. Bigelow and Pargetter, for example, present Russell's paradox of the set of all sets that are not members of themselves as strictly analogous to the puzzle of the barber who shaves all and only those who do not shave themselves.[57] Comprehension principles for sets or properties, they argue, are as mistaken as comprehension principles for barbers. The lesson they learn from this is that naturalists should proceed in a piecemeal fashion, postulating objects and properties one at a time according to theoretical need. Axioms are to be formulated to describe sets as the latter are encountered. On this view, ZF does not describe a complete universe of sets, but is rather a partial description of a slowly uncovered realm.

But we think this is to misunderstand the nature of mathematical theories and abstract objects. The abandonment of Frege's unrestricted comprehension principle[58] and its replacement by formal set theories (like ZF) is not a move from an *a priori* approach to sets to one that is *a posteriori* and of a piece with natural science. The axioms of set theory are *not* like hypotheses about a newly discovered class of fundamental particle. The pairing axiom asserts the existence of *arbitrary* pair sets using disjunction as a guide to existence; the power set axiom asserts the existence of *all* subsets of a given set. So we think Bigelow and Pargetter's analogy with empirical theories is inaccurate. The response to Russell's Paradox should not be the same as that to the Barber Paradox. Platonized Naturalists can revise their existence claims without abandoning the *a priori* status

[56]Two models of the theory of abstract objects have been developed. The first is suggested by Dana Scott, and is reported in *Abstract Objects*, *op. cit.*, Appendix A. The second is suggested by Peter Aczel and is reported in Zalta, "The Modal Object Calculus and its Interpretation," in M. de Rijke, ed., *Advances in Intensional Logic* (Dordrecht: Kluwer, forthcoming). These models describe interpretations on which both the comprehension principle for abstract objects and the comprehension principle for ordinary relations turn out true.

[57]*Science and Necessity* (Cambridge: Cambridge University Press, 1991).

[58]G. Frege, *Basic Laws of Arithmetic*, M. Furth, trans. (Berkeley: University of California Press, 1967).

of those claims and without abandoning their systematic methodology for a piecemeal one. To repeat, we can be fallibilists about the *a priori*.[59]

VII. Conclusion

We conclude with a suggestion for extending our ideas and a final observation about what we have tried to accomplish. Platonized Naturalism can be extended to account for logical objects in addition to mathematical objects. Properly understood as an *a priori* science, logic has always required objects of some sort, whether propositions, extensions, truth values, properties or pluralities (for the quantifiers of second order logic), abstract structures (for model theory), or possible worlds and other possibilia (for modal logic). Neither naturalism nor its predecessor, logical empiricism, has ever supplied a satisfactory explanation of the subject matter of logic. The only account proffered was that logic consisted of linguistic conventions. This however, as Pap showed, collapses into empirical truth.[60] There simply never was a naturalist account of logic as an object-free *a priori* science.

A Platonized Naturalist, however, can treat some logical objects (such as possible worlds, truth values, extensions, natural cardinals, and abstract models) as abstract individuals already covered by the comprehension principle.[61] The remaining logical objects require other comprehension principles, such as principles asserting a plenitude of properties, re-

[59]Given this interpretation of the development of set theory, one might wonder whether ZF itself constitutes a Principled Platonism. We rule out ZF and other set theories for the following reasons: (1) While the comprehension principles of ZF are not piecemeal, they do not assert the existence of a plenitude of sets. Just consider the many large cardinal axioms that are independent of ZF. (2) There is arbitrariness of the kind described by Benacerraf (1965) in the reductions of entities like numbers, properties, and propositions to sets. (3) Even if the reductions could be accomplished in a nonarbitrary way, reductions of intensional entities such as properties and propositions to extensional entities collapse important distinctions and so do not capture the distinguishing features of the former. So we believe that ZF offers neither a satisfactory foundation for intensional logic nor a general framework for the logical analysis of language and thought. (4) Finally, it is at least questionable whether every possible mathematical theory is reducible to ZF, and so questionable whether ZF provides the mathematics that would be used in any possible scientific theory. Hence, one cannnot successfully argue that exemplification logic and the axioms of set theory constitute the framework in which any possible natural science would be formulated.

[60]*Semantics and Necessary Truth* (New Haven: Yale University Press, 1958).

[61]See, for example, "The Theory of Fregean Logical Objects" (*op. cit.*) and "Twenty-Five Basic Theorems in Situation and World Theory" (*op. cit.*).

lations, and propositions or a plenitude of possibilia. Though we did not describe these other plenitude axioms in any detail in the present paper, they are a part of the Principled Platonism we defend.[62] They would receive the same epistemological justification as the comprehension principle for abstract individuals. Platonized Naturalism acknowledges that logic has a legimitate subject matter of its own and that this subject matter is central to the applicability of logic in the rest of natural science.

We have tried to address the traditional naturalist concerns about abstract objects. If we are right, then the bald thesis that there are no abstract objects is no longer justifiable. We have tried to develop an insight that exists at the intersection of work by Kant, Frege,[63] and Russell.[64] We defend the Kantian idea that there are synthetic *a priori* truths; we defend the Fregean idea that logic and mathematics are about objects; and we defend the Russellian idea that such objects are consistent with a robust sense of reality. We have employed rigorous logical and epistemological standards to eliminate the arbitrary, piecemeal approach to the study of abstract objects. Belief in these objects is justified if it complies with these standards.

[62]The plenum principle for possibilia is just the first-order Barcan Formula, under the actualist interpretation formulated in B. Linsky and E. Zalta, *op. cit.* Though we do not have individual, *de re* knowledge of these entities, our knowledge of the Barcan formula would nevertheless be justified on the grounds outlined in sections V and VI.

[63]*The Foundations of Arithmetic*, J. L. Austin, trans., 2nd rev. ed. (Oxford: Blackwell, 1974), and *The Basic Laws of Arithmetic*, *op. cit.*

[64]*The Problems of Philosophy* (Oxford: Oxford University Press, 1964).