Mathematical Descriptions*

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Abstract

In this paper, the authors briefly summarize how object theory uses definite descriptions to identify the denotations of the individual terms of theoretical mathematics and then further develop their object-theoretic philosophy of mathematics by showing how it has the resources to address some objections recently raised against the theory. Certain 'canonical' descriptions of object theory, which are guaranteed to denote, correctly identify mathematical objects for each mathematical theory T, independently of how well someone understands the descriptive condition. And to have a false belief about some particular mathematical object is not to have a true belief about some different mathematical object.

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In a recent paper, Buijsman (2017) argues against a component of the philosophy of mathematics that we outlined in Linsky & Zalta 1995. In particular, he raises two objections for our view about the denotation of terms in mathematical theories.[1] Briefly, our view is that the terms used in mathematical theories denote by description. Our 'object-theoretic' account of the denotation of the terms in mathematical theories is expresed by the following principle of identification, where κ_T is any primitive or well-defined singular term of theory T and $T \models p$ represents the claim (which is defined in object theory) that "In theory T, p" or "p is true in T":

$$(\vartheta)\quad \kappa_T = \imath x(A!x \,\&\, \forall F(xF \equiv T \models F\kappa_T))$$

(ϑ) says: κ_T is the abstract object that encodes exactly the properties F such that, in theory T, κ_T exemplifies F. This principle of object theory uses the notion of *encoding*, which is a primitive axiomatized by the theory and is explained not only in various publications on the theory but also well-summarized in Section 2 of Buijsman's paper. In the resulting theory, descriptions of the form $\imath x(A!x\&\forall F(xF \equiv \phi))$ are not just singular terms, but *canonical* — i.e., guaranteed to be logically proper, since the axioms of object theory guarantee that for any formula ϕ with no free xs, there is a unique abstract object that encodes exactly the properties F such that ϕ. The description that identifies κ_T in (ϑ) therefore denotes a particular abstract object, relative to the choice of κ and T.

Thus (ϑ) deploys a canonical description to identify the denotation of a well-defined term κ of theory T. Note the following features of our identification principle (ϑ):

1. The expression $T \models F\kappa_T$ (i.e., In theory T, κ_T exemplifies F) is defined in object theory to mean that theory T (which is itself identified as an abstract object that encodes propositional properties) encodes the propositional property $[\lambda y\, F\kappa_T]$, i.e., encodes the property *being such that κ_T exemplifies F*. We're not using the symbol $\models$ model-theoretically, but rather as a defined notion within object theory.

2. (ϑ) is not asserted as a definition of κ_T, as can be seen from the fact that the term appears on both sides of the identity sign. Instead

[1] This view has been defended elsewhere as well, though not exactly in the form put forward in Linsky & Zalta 1995; see Zalta 1983 (Chapter VI), Zalta 2000, and Nodelman & Zalta 2014.

(ϑ) is a principle or axiom, for which it is assumed that data of the form $T \models F\kappa_T$ is supplied by mathematical practice (see below).

3. (ϑ) identifies mathematical objects without relying on any model theory; nor are the mathematical objects covered by (ϑ) limited to those that appear in theories with only isomorphic models. Rather, object theory has its own definition of what it is to be an object of a mathematical theory: x is an object of T just in case, there is a property F such that, in T, x exemplifies F, i.e., $\exists F(T \models Fx)$.[2]

4. This account doesn't require that a mathematician be able to state or list all the theorems of a theory in order to successfully use the defining description of κ_T. Mathematicians simply need to agree that some properties and not others satisfy the formula "In theory T, κ_T exemplifies F". They can even disagree about which properties satisfy the formula — they simply need to agree that there is a body of properties that satisfy the formula. Given such a body of truths, the canonical description for κ_T is well-defined and is guaranteed to denote *independently of the epistemological state of the mathematician*. As long as the person uses the name of T to identify the theory in question, the causal chain of reference traces back to the first use of the name 'T' to introduce the theory. This doesn't depend on the knowledge or beliefs of the person using the name.

To see that there is content to the description used for the identification of mathematical objects, note that formulas of the form $T \models F\kappa_T$ become analytic truths when imported into object theory as part of its analysis of T. Specifically, the theorems of T, i.e., claims of the form $T \vdash \phi$, are imported into object theory as analytic truths of the form $T \models \phi$, where the terms of ϕ are indexed to the theory T. So, for example, ZF $\vdash \emptyset \in \{\emptyset\}$ becomes imported into object theory as the claim ZF $\models \emptyset_{\mathrm{ZF}} \in \{\emptyset\}_{\mathrm{ZF}}$, which asserts that, in ZF, the empty set is an element of the unit set of the empty set. (Higher-order object theory is used to give a similar analysis of mathematical relations such as $\in_{\mathrm{ZF}}$, but for the purposes of this paper, we shall not discuss this further application of the theory.)

[2] A somewhat more refined definition was put forward in Nodelman & Zalta 2014, to avoid a concern about indiscernibles: x is an *object of* a theory T just in case x is distinguishable in T, i.e., $T \models \forall y(y \neq_T x \rightarrow \exists F(Fx \,\&\, \neg Fy))$.

Object theory thus identifies mathematical objects without invoking any model theory. It treats model-theory simply as applied mathematics (indeed, applied set theory) and so the terms of model theory become subject to the object-theoretic analysis. Our analysis therefore presupposes no mathematical primitives, but rather uses classical logic extended by the notion of xF (x encodes F).

Though Buijsman raises several concerns for our view, there are two main objections. Whereas Buijsman believes his objections apply to any view that invokes a plenitude of Platonic objects (i.e., "one on which all possible mathematical objects exist, roughly speaking"), he focuses on our view as a representative example. It is not clear to us, however, that other such theories will have the resources that we use below to respond. So our response will not necessarily apply to other, similar theories of reference.

The first objection occurs in the following passage (2017, 132):

> This raises the question what exactly is needed successfully to employ the definite descriptions to refer to mathematical objects. Linsky and Zalta say that all one needs is to understand the descriptive condition, but that leaves several options open. A very strong interpretation would be that one needs to know precisely which properties are encoded by the object one is trying to refer to. ... Another, weaker, interpretation is to view this as saying that one needs to understand that the object referred to encodes all of the properties that a specific theory (which one has in mind) attributes to that object. In this case, all one would need to know is which theory the object is supposed to belong to, and what place it has in that theory. ... The even weaker interpretation, where one would only need to know which theory the object is a part of, seems to be too weak. For if one wants to refer to a specific object, one will have to know enough that one actually succeeds in picking out a particular object.

None of these interpretations are correct. Given the truth of the comprehension and identity principles for abstract objects, it is a theorem that $\exists!x(A!x \,\&\, \forall F(xF \equiv \phi))$ and, hence, that descriptions of the form $\imath x(A!x \,\&\, \forall F(xF \equiv \phi))$ always denote. So, in a particular case, where we represent the null set of ZF as $\emptyset_{\mathrm{ZF}}$ and represent claims of the form "In ZF, the null set is F" as ZF $\models F\emptyset_{\mathrm{ZF}}$, the theory we proposed guarantees that the defining description,

$$\imath x(A!x \,\&\, \forall F(xF \equiv \mathrm{ZF} \models F\emptyset_{\mathrm{ZF}}))$$

denotes an object. So all we as metaphysicians have to do to refer to that object is to *understand the meaning* of the description, i.e., understand that descriptions like the above, using technical terms such as *encoding* and *truth in a theory*, are guaranteed to denote the unique object that satisfies the matrix.[3] This can be spelled out, formally if need be, in terms of existence and identity claims.[4] Buijsman supposes we have to be *acquainted* with all the properties F such that in ZF, $\emptyset_{\mathrm{ZF}}$ exemplifies F. We don't.

We think our view is consistent with the following suggestion in Benacerraf 1981 (42–43):

> But in reply to Kant, logicists claimed that these propositions are a priori because they are analytic—because they are true (false) merely "in virtue of" the meanings of the terms in which they are cast. Thus to know their meanings is to know all that is required for a knowledge of their truth. No empirical investigation is needed. The philosophical point of establishing the view was nakedly epistemological: logicism, if it could be established, would show that our knowledge of mathematics could be accounted for by whatever would account for our knowledge of language. And, of course, it was assumed that knowledge of language could *itself* be accounted for in ways consistent with empiricist principles, that language was itself entirely learned. Thus, following Hume, all our knowledge could once more be seen as concerning either "relations of ideas" (analytic and a priori) or "matters of fact".[5]

Clearly, we don't have to provide some empiricist theory of how we understand language — that is something we as metaphysicians get to presuppose. All we have to account for is the well-definedness of the definite descriptions used to identify mathematical objects. For if they are well-defined, our ability to use and understand language secures the reference, since the descriptions are guaranteed to denote. Of course, Buijsman can raise epistemological questions about how we understand language, and how we know what someone is referring to, but our view sidelines those questions by starting with the fact that we understand mathematical theories and can refer to them by name.

The second objection Buijsman raises is that, on our theory, to hold a false belief about a mathematical object is to hold a true belief about some different mathematical object (Buijsman 2017, 134). He says:

> Reference, then, is unstable on Linsky's and Zalta's view, because it seems that as soon as one believes that a mathematical object has some property (that the original object did not encode), one will be referring to another mathematical object.

But this objection fails to distinguish the *sense* of a mathematical term from its denotation. Someone can use an expression to denote an object even though she has only false beliefs about that object. Person X might see "Dr. Lauben – General Practitioner – 8am–5pm" on a sign in front of a building, and come to hold all sorts of beliefs about Lauben. But unbeknownst to X, the day before, Lauben was stripped of his medical license, and has ceased to practice at the building in question. X can nevertheless refer to Lauben, even though most of her beliefs about Lauben are false. Those beliefs are connected with the *sense* of the term 'Lauben', while the denotation of the term is secured by the causal chain of reference.

We say that the very same thing applies to the case of mathematical terms. Consider Buijsman's discussion on pp. 134–5, about the number one of PA (1_{PA}). In object theory, we identify that object as:

$$1_{\mathrm{PA}} = \imath x(A!x \,\&\, \forall F(xF \equiv \mathrm{PA} \models F1_{\mathrm{PA}}))$$

[3] At the end of the present paper, we offer one final point in response to anyone who objects that our analysis requires a technical understanding of object theory to successfully refer using a mathematical term.

[4] Formally, the semantics of descriptions can be spelled out as follows. Where $\mathcal{I}$ is an interpretation of the language, f is an assignment function, $\boldsymbol{d}_{\mathcal{I},f}(\tau)$ is the denotation of term τ w.r.t $\mathcal{I}$ and f, and $[\phi]_{\mathcal{I},f}$ asserts ϕ is true w.r.t. $\mathcal{I}$ and f, we simply add a recursive clause that says:

$$\boldsymbol{d}_{\mathcal{I},f}(\imath x\phi) = \begin{cases} \boldsymbol{o} \text{ if } f(x)\!=\!\boldsymbol{o} \,\&\, [\phi]_{\mathcal{I},f} \,\&\, \forall \boldsymbol{o}'\forall f'(f'(x)\!=\!\boldsymbol{o}' \,\&\, [\phi]_{\mathcal{I},f'} \rightarrow \boldsymbol{o}'\!=\!\boldsymbol{o}) \\ \text{undefined, otherwise} \end{cases}$$

This says that the denotation of $\imath x\phi$ is the object $\boldsymbol{o}$ if and only if there is an assignment f such (a) that f assigns the object o to the variable x, (b) ϕ is true w.r.t. $\mathcal{I}$ and f, and (c) for all objects o' and assignments f', if f' assigns $\boldsymbol{o}'$ to x and ϕ is true w.r.t. $\mathcal{I}$ and f', then $\boldsymbol{o}'$ is identical to $\boldsymbol{o}$. This captures Russell's 1905 analysis of descriptions semantically.

[5] We recognize that this passage comes in the context where Benacerraf is presenting the "myth he learned as a youth", but this bit is not the mythical part!

Now Buijsman's first case of supposed reference failure concerns someone who holds the false belief that the above number is prime. Buijsman then concludes (2017, 135):

> ... suppose that the agent retains all of her old beliefs about the number one, but mistakenly comes to believe that the number one is a prime number. In that case, she will try to refer to the number that encodes all of the propositions she believed in the first case, except that now the number encodes that one is a prime number. Importantly, the number she comes to refer to does still encode that the standard definition of prime numbers applies to it. Thus, since encoding is closed under entailment, the number she refers to will also encode that one is not a prime number. As a result, this one mistake has resulted in reference to a number which belongs to an inconsistent system — one where it is true that one is a prime number and that one is not a prime number (note that there is indeed such a mathematical object on Linsky's and Zalta's view).

The claims in this passage are inaccurate. First, the phrases "the number that encodes all of the propositions she believed", "except that now the number encodes that one is a prime number", and others in this passage, are not quite what the theory says. The number one of PA, on our view, encodes properties of numbers not propositions, whereas the theory PA encodes propositions by encoding propositional properties.

Second, we have to be clear: the theory PA encodes only the propositions that are the theorems of PA. The fact that someone holds the false belief that the number one of PA is prime doesn't imply that the theory itself "encodes that one is a prime number". Second, given that the theory is unaffected by what someone believes, 1_{PA} does not encode the property of being prime. Third, it doesn't follow from the fact that someone falsely believes that 1_{PA} is prime, that 1_{PA} encodes being prime — rather, 1_{PA} denotes the object described above. The denotation of the term hasn't changed because of someone's false belief; all one can conclude is that the person associates a *sense* of the term 1_{PA} that involves the property of being prime.

Indeed, object theory has been used to identify the senses of terms in natural language (Zalta 1988, 2001). For example, the senses of the terms 'Mark Twain' and 'Samuel Clemens' can be assigned different abstract objects, even though both names denote the same object. In the case of 'Lauben' described above, you could say that the sense of 'Lauben' for person X is an abstract object that encodes *misinformation*, since X *believes* Lauben is a doctor and so the sense of the term 'Lauben' encodes being a doctor. But this doesn't imply the denotation of 'Lauben' is something that is a doctor. In the object-theoretic reconstruction of Fregean senses, the Fregean sense of a term need not *determine* the referent.

These reflections lend themselves to a natural response to the following passage in Buijsman (2017, 136):

> Presumably, we can legitimately disagree about whether or not something holds of a mathematical object. This will be difficult, however, if we are thinking about different mathematical objects — something which may well be the case, as a change in ascribed properties implies a change in the denoted object. So it is hard to see that we legitimately disagree about the properties of a mathematical object, as such disagreement may well imply that we are actually talking about different objects.

We claim that disagreement doesn't imply that we are talking about different objects. First, assume person Y says, in the context of ZF, that $\aleph_0$ can be put into one-to-one correspondence with $2^{\aleph_0}$, and person Z (correctly) disagrees. How does it follow that they are talking about different objects? Our theory says that Y and Z are both referring to $\aleph_{0\mathrm{ZF}}$ and $2^{\aleph_0}_{\mathrm{ZF}}$. Why should we accept, as Buijsman claims, that Y and Z are thinking about different mathematical objects, or accept that they can't legimately disagree without talking about different objects? In this particular case, we would conclude that Y mistakenly believes that it is a theorem of ZF that $\aleph_0$ can be put into one-to-one correspondence with $2^{\aleph_0}$. Y can even learn of this error without understanding the proof of the power set theorem, simply from the testimony of an authority, perhaps even Z.

We may similarly respond to the following passage (Buijsman 2017, 136):

> Testimony will also be very difficult because testimony is supposed to be a transfer of knowledge from the speaker to the hearer. As long as testimony is considered to be such a transfer of knowledge, or justified beliefs, the instability of reference will make testimony in the case of mathematics very difficult. For if two people ascribe

> slightly different properties to a mathematical object, then they will be talking about different mathematical objects.

The last line of this passage doesn't correctly describe our theory. As we've already seen, the fact that two people ascribe different properties to a mathematical object doesn't imply that they are talking about different mathematical objects. At least one of them has made a mistake.

It should be noted here that in the special case of fictions and mathematical objects, object theory provides not only the denotations of the terms but also their senses. The denotations are tied to the theory and, in particular, to the mathematical practice of supposing that theories don't change even though people have false beliefs about them. So, 1_{PA} denotes the object described above notwithstanding the false beliefs of the person in Buijsman's case. However, the *sense* of the expression 1_{PA} will be an abstract object that encodes the property *being prime* for the person who has the false belief.

In this case, we rely on the fact that reference traces back by way of causal chains. In ordinary (i.e., non-mathematical and non-fictional cases), the beginning of the chain is a baptism of an ordinary object, whereas in mathematical cases, the beginning of the chain is a *theory*, in the context of which unique abstract objects are denoted, given our analysis. Instead of a baptism of a concrete object, someone *authors* a theory, and given that theory, our analysis identifies the relevant objects (Zalta 2003).

The other cases that Buijsman describes on p. 134 all fail for the same reason — in each case, the false beliefs of the person in question have no effect on what object is denoted by the expression 1_{PA}. They may in fact have *only* false beliefs about 1_{PA}, yet they still have beliefs about *it*. Those false beliefs are captured by the fact that their sense of the expression 1_{PA} encodes properties that 1_{PA} doesn't encode.

Though we've now answered the two main objections Buijsman raises, we conclude with discussion of a few other objections that he includes in the paper.

Buijsman suggests that our view of mathematical reference requires that mathematical theories have "isomorphic models" (p. 133) and implies that only objects that are *definable* by a theory in the model-theoretic sense can be referred to. He seems to presuppose that we need model theory to give an account of reference. But from our point of view, that would be circular, since model theory is just applied set theory. Our

analysis, as noted previously, also applies to the reference of the terms of model theory. Thus, we are offering a mathematics free account of mathematical objects, mathematical reference, and mathematical truth. We don't suppose that an object of a mathematical theory is something over which the bound variables of the theory range, but rather define the objects of a theory to be those that have properties according to the theory (modulo footnote 2). For us, a mathematical theory is itself an abstract object identified by the propositions it encodes — ZF is the abstract object that encodes just the properties F of the form $[\lambda y\ p]$ which are such that p is true in ZF, where the truths are identified by mathematical practice, i.e., by the theorems of the theory. So to give this objection, Buijsman has to use a notion of "definability" that is different from the one that we have in mind, and indeed one that is out of keeping with our view of mathematical theories as abstract objects that encode propositions.

The next issue to discuss is the question of how laypersons (non-mathematicians) learn mathematics (e.g., by testimony from experts), and how they come to refer to mathematical objects in their discourse. If Buijsman's paper is about how laypersons refer to mathematical objects, then one should focus on (a) how the use of a mathematical term by a layperson is acquired through a causal chain that traces back to the community of mathematicians and (b) how the use of the term by the person in question thereby *depends on* the use of the term by mathematicians. In both of these cases (a) and (b), the reference is given by the relevant instance of (ϑ), and this is consistent with the person having false beliefs about the mathematical terms in question. Again, their false beliefs or ignorance about the theories in question indicate only that their *sense* of the mathematical term in question encodes properties that aren't encoded by the denotation of the term. Just as with mathematical experts, such cases of false belief and ignorance don't imply that the term is being used to denote some object other than the one given by the relevant instance of (ϑ).

Finally, one might suppose that Buijsman is objecting that we are requiring that both mathematicians and non-mathematicians alike know some object theory in order to understand the canonical descriptions that provide an analysis of the denotations of mathematical terms. But there are two points to make about this.

First, one *does* have to understand object theory to understand our view of what mathematical terms denote. Buijsman, and others who read our proposals carefully, clearly understand the technical descriptions involving the notion of *encoding*; they wouldn't be able to raise cogent objections if they didn't understand our view.

Second, our theory doesn't require that anyone understand object theory in order to successfully refer to a mathematical object. To see this, let us put aside cases of *derivative* or *dependent* reference, of the kind discussed three paragraphs back. In those cases, the person in question uses a mathematical term κ with the intention to refer to whatever the mathematicians refer to. Let's focus instead on cases of *non-derivative* reference, in which the person in question has some understanding or knowledge of T. In these cases, the object-theoretic analysis of mathematics doesn't place any special conditions on reference. For example, the theory is compatible with saying: if T is a consistent theory and S knows only that κ is a well-defined term of T, then S can use κ to successfully refer. This does *not* require that S be able to give a T-based description of what κ denotes. If just these minimal conditions are satisfied, then the object-theoretic analysis gives a theoretical description (in purely metaphysical terms) of what object is being referred to. This doesn't require that S have any knowledge of object theory. Knowledge of the latter isn't required for mathematical knowledge; it only serves to tell us, in theoretical metaphysical terms, what the objects of mathematical theories are.

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